

Logarithmic stabilization of the Euler-Bernoulli transmission plate equation with locally distributed Kelvin-Voigt damping

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Abstract

In this paper we will study the asymptotic behaviour of the energy decay of a transmission plate equation with locally distributed Kelvin-Voigt feedback. Precisely, we shall prove that the energy decay at least logarithmically over the time. The originality of this method comes from the fact that using a Carleman estimate for a transmission second order system which will be derived from the plate equation to establish a resolvent estimate which provide, by the famous Burq's result [Bur98], the kind of decay mentioned above.

Key words and phrases: Transmission problem, Kelvin-Voigt damping, Euler-Bernoulli plate equation, energy decay, Carleman estimates.

Mathematics Subject Classification: 35A01, 35A02, 35M33, 93D20.

1 Introduction and statement of results

In recent years, there has been much interest in the stability problems for elastic systems with locally distributed damping. Most of the works were devoted to the viscous damping, i.e., the damping is proportional to the velocity (see for instance [CFNS91] and [Zua90]). Structures with local viscoelasticity arise from use of smart material or passive stabilization of structures. However, very little is known about exponential stability for elastic systems with local viscoelastic damping, although there is a fairly deep understanding when the damping is distributed over the entire domain but only for 1-dimension (see [LL02]). To our knowledge, the first paper in this direction was published in 1998 by Liu and Liu [LL98] where they obtained exponential stability for the Euler-Bernoulli beam equation with local Kelvin-Voigt damping. Noting that in our knowledge there are zero results at least for the multi-dimension Euler-Bernoulli plate equation case.

Consider a clamped elastic domain in \mathbb{R}^n , ($n \geq 2$) which is made of a viscoelastic material with Kelvin-Voigt constitutive relation in which a transmission effect has been established such a way that the damping is locally effective in only one side the transmission boundary. By the Kirchhoff hypothesis, neglecting the rotatory inertia, the transversal vibration (see [CLL98] for the modeling problem) can be described as follows: Let Ω and Ω_1 be two open, bounded and connected domains with smooth boundary respectively Γ and S such that $\Omega_1 \subset \Omega$ and

$\overline{S} \cap \overline{\Gamma} = \emptyset$. We set also $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ which is an open connected domain with boundary $\partial\Omega_2 = \Gamma \cup S$.

We are going to study the following transmission and boundary value problem

$$(1.1) \quad \begin{cases} \partial_t^2 u_1 + \Delta(c_1^2 \Delta u_1 + a \Delta \partial_t u_1) = 0 & \text{in } \Omega_1 \times]0, +\infty[, \\ \partial_t^2 u_2 + c_2^2 \Delta^2 u_2 = 0 & \text{in } \Omega_2 \times]0, +\infty[, \\ u_1 = u_2 & \text{on } S \times]0, +\infty[, \\ \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \times]0, +\infty[, \\ c_1 \Delta u_1 = c_2 \Delta u_2 & \text{on } S \times]0, +\infty[, \\ c_1 \partial_\nu \Delta u_1 = c_2 \partial_\nu \Delta u_2 & \text{on } S \times]0, +\infty[, \\ u_1 = 0 & \text{on } \Gamma \times]0, +\infty[, \\ \Delta u_1 = 0 & \text{on } \Gamma \times]0, +\infty[, \\ u_1(x, 0) = u_1^0(x), \partial_t u_1(x, 0) = u_1^1(x) & \text{in } \Omega_1, \\ u_2(x, 0) = u_2^0(x), \partial_t u_2(x, 0) = u_2^1(x) & \text{in } \Omega_2. \end{cases}$$

Where ∂_ν denotes the unit outward normal vector of Ω_1 and Ω respectively in S and Γ , c_1, c_2 are strictly positives constants and a is a non negative bounded functions in Ω_1 and we suppose that a vanishing near the boundary S such that there exist a non empty open domain $\omega \subset \Omega_1$ such that a is strictly positives in $\overline{\omega}$.

The energy of a solution of (1.1) at time $t \geq 0$ is defined by

$$E(t) = \frac{1}{2} \int_{\Omega_1} \left(|\partial_t u_1(x, t)|^2 + c_1^2 |\Delta u_1(x, t)|^2 \right) c_1^{-1} dx + \frac{1}{2} \int_{\Omega_2} \left(|\partial_t u_2(x, t)|^2 + c_2^2 |\Delta u_2(x, t)|^2 \right) c_2^{-1} dx.$$

By Green's formula we can prove that for all $t_1, t_2 > 0$ we have

$$E(t_2) - E(t_1) = -c_1^{-1} \int_{t_1}^{t_2} \int_{\Omega_1} a |\Delta \partial_t u_1(x, t)|^2 dx dt,$$

and this mean that the energy is decreasing over the time.

We define the operator \mathcal{A} by

$$\mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = (v_1, v_2, -\Delta(c_1^2 \Delta u_1 + a \Delta v_1), -c_2^2 \Delta^2 u_2)$$

in the Hilbert space $\mathcal{H} = X \times H$ where $H = H_1 \times H_2 = L^2(\Omega_1, c_1^{-1} dx) \times L^2(\Omega_2, c_1^{-1} dx)$ and

$$(1.2) \quad X = \{(u_1, u_2) \in H : u_1 \in H^2(\Omega_1), u_2 \in H^2(\Omega_2), u_2|_\Gamma = 0, u_1|_S = u_2|_S, \partial_\nu u_1|_S = \partial_\nu u_2|_S\},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{(u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2, \Delta(c_1^2 \Delta u_1 + a \Delta v_1), c_2^2 \Delta^2 u_2) \in \mathcal{H}, \Delta u_2|_\Gamma = 0, c_1 \Delta u_1|_S = c_2 \Delta u_2|_S, c_1 \partial_\nu \Delta u_1|_S = c_2 \partial_\nu \Delta u_2|_S\}.$$

Now we are able to state our main results

Theorem 1.1 *There exists $C > 0$ such that for every $\mu \in \mathbb{R}$ with $|\mu|$ large, we have*

$$(1.3) \quad \|(\mathcal{A} - i\mu \text{Id})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{C|\mu|}.$$

As an immediate consequence of the previous theorem (see [Bur98] and more recently [BD08]), we get the following rate of decrease of energy

Theorem 1.2 *For any $k \in \mathbb{N}$, there exists a constant $C > 0$ such that for any initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}^k)$, the energy $E(t)$ of the system (1.1) whose solution $u(x, t)$ is starting from $(u_1^0, u_2^0, u_1^1, u_2^1)$ satisfy*

$$E(t) \leq \frac{C}{(\ln(2+t))^{2k}} \|(u_1^0, u_2^0, u_1^1, u_2^1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \quad \forall t > 0.$$

Remarks 1.1

- 1) Under one assumption to the coefficients c_1 and c_2 , Ammari and Vodev [AV09] have proved an exponential stabilization result for the Euler-Bernoulli transmission plate equation with boundary dissipation. Again for a transmission model, Ammari and Nicaise [AN10] have proved, under some geometric condition, an exponential stabilization for a coupled damped wave equation with a damped Kirchhoff plate equation.
- 2) To prove Theorem 1.1 and Theorem 1.2, we make use the Carleman estimates to obtain information about the resolvent in a boundary domain, the cost is to use phases functions satisfying Hörmander's assumption. Albano [Alb00] proved a Carleman estimate for the plate operator, by decomposing the operator as the product of two Schrödinger ones and gives for each of them the corresponding Carleman estimate then by making together these two estimates we obtain the result. But here we will not need to have a Carleman estimate for the plate equation, namely inspiring from the Albano's decomposition we will derive a second order transmission system to which we are going to apply an appropriate Carleman estimates (see section 3) for a suitable phases functions, thus we will obtain the resolvent estimate of Theorem 1.1.
- 3) Theorem 1.1 and Theorem 1.2 are analogous to those of Fathallah [Fat11], in the case of hyperbolic-parabolic coupled system, and Lebeau and Robbiano [LR97] results, in the case of scalar wave equation without transmission, but our method is different from their because it consist to use the Carleman estimates directly for the stationary operator without going through the interpolation inequality.
- 4) For various purposes, several authors have focused to the transmission problems where they needed to find a Carleman estimates near the interface, such as the works of Bellassoued [Bel03] and Fathallah [Fat11] for the stabilization problems and that also of Le Rousseau and Robbiano [RR10] for a control problem.
- 5) Note that in the case where it has no transmission of the problem (1.1), Theorem 1.1 and Theorem 1.2 remain valid and in this case we need only the classical Carleman estimates (see [LR97] and [LR95]).

In this paper C will always be a generic positive constant whose value may be different from one line to another.

The outline of this paper is as follow. In section 2 we prove the well-Posedness of the problem (1.1), in section 3 we give a global Carleman estimate and we will construct a suitable phases functions and in section 4 we prove the resolvent estimate given by Theorem 1.1.

2 Well-Posedness of the problem

To prove the Well-Posedness of the problem (1.1) we are going to use the semigroups theory. Our strategy consist to write the equations as a Cauchy problem with an operator which generates a semigroup of contractions.

Throughout this paper, we denote the inner product in the space $H = H_1 \times H_2$ by

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_H = \int_{\Omega_1} u_1(x) \overline{v_1(x)} c_1^{-1} dx + \int_{\Omega_2} u_2(x) \overline{v_2(x)} c_2^{-1} dx,$$

The Cauchy problem is written in the following form

$$\begin{cases} \partial_t \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} (t) = \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} (t) & t \in]0, +\infty[, \\ \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} (0) = \begin{pmatrix} u_1^0 \\ u_2^0 \\ v_1^1 \\ v_2^1 \end{pmatrix}. \end{cases}$$

Now we have to specify the functional space and the domain of the operator \mathcal{A} . In the space H we define the operator G by

$$G \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (-c_1 \Delta u_1, -c_2 \Delta u_2) \quad \forall (u_1, u_2) \in \mathcal{D}(G)$$

with domain $\mathcal{D}(G) = X$ defined in (1.2). The space X is equipped with the norm

$$\|(u_1, u_2)\|_X = \|G(u_1, u_2)\|_H$$

and we defined the graph norm of G by

$$\|(u_1, u_2)\|_{gr(G)}^2 = \|(u_1, u_2)\|_H^2 + \|G(u_1, u_2)\|_H^2$$

then we have the following

Proposition 2.1 $(X, \|\cdot\|_X)$ is a Hilbert space with a norm equivalent to the graph norm of G .

Proof :

It is well known that if G is a closed operator then $(X, \|\cdot\|_{gr(G)})$ is a Hilbert space. Thus to prove the proposition it suffices to show that G is closed and both norms are equivalent.

By Green's formula and Poincaré inequality it is easy to show that there exists $C > 0$ such that

$$\left\langle G \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_H = \|\nabla u_1\|_{L^2(\Omega_1)}^2 + \|\nabla u_2\|_{L^2(\Omega_2)}^2 \geq C \|(u_1, u_2)\|_H^2 \quad \forall (u_1, u_2) \in X.$$

Then G is a strictly positive operator and we have

$$\|G(u_1, u_2)\|_H \cdot \|(u_1, u_2)\|_H \geq \left\langle G \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_H \geq C \|(u_1, u_2)\|_H^2 \quad \forall (u_1, u_2) \in X$$

which prove the equivalence between the two norms.

Now since G is positive then by in [TW09, Proposition 3.3.5], $-G$ is m-dissipative and thus G is a closed operator. This completes the proof. \blacksquare

This last result allows us to properly define the functional space of the operator \mathcal{A} .

Proposition 2.2 *The two spaces $(X, \|\cdot\|_2)$ and $(X, \|\cdot\|_X)$ are algebraically and topologically the same. Where we have defined $\|\cdot\|_2$ by*

$$\|(u_1, u_2)\|_2^2 = \|u_1\|_{H^2(\Omega_1)}^2 + \|u_2\|_{H^2(\Omega_2)}^2, \quad \forall (u_1, u_2) \in X.$$

Proof :

We have only to prove that the two norms are equivalent.

First, we note that $(X, \|\cdot\|_2)$ is a Hilbert space because X is a closed subspace of $H^2(\Omega_1 \cup \Omega_2)$, in addition we have

$$\|(u_1, u_2)\|_X^2 = \|\Delta u_1\|_{L^2(\Omega_1)}^2 + \|\Delta u_2\|_{L^2(\Omega_2)}^2 \leq C \|(u_1, u_2)\|_2^2 \quad \forall u \in X,$$

and while $(X, \|\cdot\|_X)$ is also a Hilbert space, then according to the Banach theorem (see [EMT04, Corollary 9.2.3]) the two norms are equivalent. \blacksquare

We set $\mathcal{H} = X \times H$ the Hilbert space with the norm

$$\|(u_1, u_2, v_1, v_2)\|^2 = \|(u_1, u_2)\|_X^2 + \|(v_1, v_2)\|_H^2 \quad \forall (u_1, u_2, v_1, v_2) \in \mathcal{H},$$

and we recall that the domain of the operator \mathcal{A} is defined by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{ (u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2, \Delta(c_1^2 \Delta u_1 + a \Delta v_1), c_2^2 \Delta^2 u_2) \in \mathcal{H}, \Delta u_2|_\Gamma = 0, \\ c_1 \Delta u_1|_S = c_2 \Delta u_2|_S, c_1 \partial_\nu \Delta u_1|_S = c_2 \partial_\nu \Delta u_2|_S \}. \end{aligned}$$

Theorem 2.1 *Under the above assumptions, the operator \mathcal{A} is m-dissipative and especially it generates a strongly semigroup of contractions in \mathcal{H} .*

Proof :

According to Lumer-Phillips theorem (see for exemple [TW09, p.103]) we have only to prove that \mathcal{A} is m-dissipative.

Let $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ then by Green's formula we have

$$\begin{aligned} \operatorname{Re} \left\langle \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} &= \operatorname{Re} \left\langle \begin{pmatrix} v_1 \\ v_2 \\ -\Delta(c_1^2 \Delta u_1 + a \Delta v_1) \\ -c_2^2 \Delta^2 u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= -c_1 \|a^{\frac{1}{2}} \Delta v_1\|_{L^2(\Omega_1)}^2 \leq 0. \end{aligned}$$

This shows that \mathcal{A} is dissipative.

Let now $(f_1, f_2, g_1, g_2) \in \mathcal{H}$ and our purpose is to find a couple $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ such that

$$(\operatorname{Id} - \mathcal{A}) \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ v_1 + \Delta(c_1^2 \Delta u_1 + a \Delta v_1) \\ v_2 + c_2^2 \Delta^2 u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix}$$

more explicitly we have to find $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ such that

$$\begin{cases} v_1 = u_1 - f_1 \\ v_2 = u_2 - f_2 \\ u_1 + \Delta((c_1^2 + a)\Delta u_1 - a\Delta f_1) = f_1 + g_1 \\ u_2 + c_2^2 \Delta^2 u_2 = f_2 + g_2. \end{cases}$$

First note that, by Riesz representation theorem, there exists a unique $(u_1, u_2) \in X = \mathcal{D}(G)$ such that for all $(\varphi_1, \varphi_2) \in X$ we have

$$(2.1) \quad \begin{aligned} &\langle f_1 + g_1, \varphi_1 \rangle_{L^2(\Omega_1)} + \langle f_2 + g_2, \varphi_2 \rangle_{L^2(\Omega_2)} + \langle a\Delta f_1, \Delta \varphi_1 \rangle_{L^2(\Omega_1)} = \langle u_1, \varphi_1 \rangle_{L^2(\Omega_1)} \\ &\quad + \langle u_2, \varphi_2 \rangle_{L^2(\Omega_2)} + \langle (c_1^2 + a)\Delta u_1, \Delta \varphi_1 \rangle_{L^2(\Omega_1)} + c_2^2 \langle \Delta u_2, \Delta \varphi_2 \rangle_{L^2(\Omega_2)}. \end{aligned}$$

In particular for all $(\varphi_1, \varphi_2) \in \mathcal{C}_c^\infty(\Omega_1) \times \mathcal{C}_c^\infty(\Omega_2)$ the expression (2.1) yields

$$\begin{aligned} &\langle \Delta((c_1^2 + a)\Delta u_1 - a\Delta f_1) + (u_1 - f_1 - g_1), \varphi_1 \rangle_{D'(\Omega_1)} = 0, \\ &\langle c_2^2 \Delta^2 u_2 + (u_2 - f_2 - g_2), \varphi_2 \rangle_{D'(\Omega_2)} = 0. \end{aligned}$$

then we obtain

$$(2.2) \quad \begin{aligned} u_1 + \Delta((c_1^2 + a)\Delta u_1 - a\Delta f_1) &= f_1 + g_1 \quad \text{in } L^2(\Omega_1), \\ u_2 + c_2^2 \Delta^2 u_2 &= f_2 + g_2 \quad \text{in } L^2(\Omega_2). \end{aligned}$$

Now if we return again to the expression (2.1) then by Green's formula we can write it as follows

$$\begin{aligned} &\langle \Delta((c_1^2 + a)\Delta u_1 - a\Delta f_1) + (u_1 - f_1 - g_1), \varphi_1 \rangle_{L^2(\Omega_1)} + \langle c_2^2 \Delta^2 u_2 + (u_2 - f_2 - g_2), \varphi_2 \rangle_{L^2(\Omega_2)} \\ &= -\langle c_2 \Delta u_2, \partial_\nu \varphi_2 \rangle_{L^2(\Gamma)} - \langle c_1 \Delta u_1, \partial_\nu \varphi_1 \rangle_{L^2(S)} + \langle c_2 \Delta u_2, \partial_\nu \varphi_1 \rangle_{L^2(S)} \\ &\quad + \langle c_1 \partial_\nu \Delta u_1, \varphi_2 \rangle_{L^2(S)} - \langle c_2 \partial_\nu \Delta u_2, \varphi_2 \rangle_{L^2(S)}. \end{aligned}$$

then by (2.2) we get for all $(\varphi_1, \varphi_2) \in X$ that

$$\langle c_1 \partial_\nu \Delta u_1 - c_2 \partial_\nu \Delta u_2, \varphi_1 \rangle_{L^2(S)} - \langle c_1 \Delta u_1 - c_2 \Delta u_2, \partial_\nu \varphi_1 \rangle_{L^2(S)} - \langle c_2 \Delta u_2, \partial_\nu \varphi_2 \rangle_{L^2(\Gamma)} = 0,$$

which yields the following equalities

$$c_1 \Delta u_1|_S = c_2 \Delta u_2|_S, \quad c_1 \partial_\nu \Delta u_1|_S = c_2 \partial_\nu \Delta u_2|_S, \quad \Delta u_2|_\Gamma = 0.$$

And this concludes the proof. ■

One consequence of this last result is that if we assume that $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A})$, there exists a unique solution of (1.1) which can be expressed by means of a semigroup on \mathcal{H} as follows

$$(2.3) \quad \begin{pmatrix} u_1 \\ u_2 \\ \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} = e^{tA} \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{pmatrix}$$

where $e^{t\mathcal{A}}$ is the C_0 -semigroup of contractions generated by the operator \mathcal{A} . And we have the following regularity of the solution

$$\begin{pmatrix} u_1 \\ u_2 \\ \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} \in C([0, +\infty[, \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty[, \mathcal{H}).$$

And if $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{H}$, the function $(u_1(t), u_2(t))$ given by (2.3) is the mild solution of (1.1) and it lives in $C([0, +\infty[, \mathcal{H})$.

3 Carleman estimate and construction weight functions

3.1 Carleman estimate

We consider two open and disjoint domains \mathcal{O}_1 and \mathcal{O}_2 in which we define respectively the second order elliptic semi-classical operators $P_1 = -h^2\Delta - \alpha_1 h$ and $P_2 = -h^2\Delta - \alpha_2 h$ with principal symbol $p(x, \xi) = |\xi|^2$ where h is a very small semi-classical parameter and $\alpha_1, \alpha_2 \in \mathbb{R}$, and we suppose that $\partial\mathcal{O}_1 = \gamma \cup \gamma_1$, $\partial\mathcal{O}_2 = \gamma \cup \gamma_2$ and $\overline{\gamma}_1 \cap \overline{\gamma}_0 = \overline{\gamma}_2 \cap \overline{\gamma}_0 = \emptyset$.

Let $\varphi_1 \in \mathcal{C}^\infty(\overline{\mathcal{O}}_1)$ and $\varphi_2 \in \mathcal{C}^\infty(\overline{\mathcal{O}}_2)$ two real value functions. We define the two adjoint operators $P_{\varphi_1} = e^{\varphi_1/h} P_1 e^{\varphi_2/h}$ and $P_{\varphi_2} = e^{\varphi_1/h} P_2 e^{\varphi_2/h}$ of principal symbol respectively $p_1(x, \xi) = p(x, \xi + i\nabla\varphi_1)$ and $p_2(x, \xi) = p(x, \xi + i\nabla\varphi_2)$.

By denoting ∂_ν the unit outward normal vector of \mathcal{O}_1 and \mathcal{O}_2 respectively in $\gamma \cup \gamma_1$ and γ_2 we assume that the weight function φ_1 and φ_2 satisfies

- 1) $|\nabla\varphi_1|(x) > 0, \forall x \in \overline{\mathcal{O}}_1$ and $|\nabla\varphi_2|(x) > 0, \forall x \in \overline{\mathcal{O}}_2$,
- 2) $\partial_\nu\varphi_1|_{\gamma_1} \neq 0$ and $\partial_\nu\varphi_2|_{\gamma_2} < 0$,
- 3) $\varphi_1|_\gamma = \varphi_2|_\gamma$,
- 4) $(\partial_\nu\varphi_1)|_\gamma < 0, (\partial_\nu\varphi_2)|_\gamma < 0$ and $(\partial_\nu\varphi_1)|_\gamma^2 - (\partial_\nu\varphi_2)|_\gamma^2 > 0$,
- 5) The sub-ellipticity condition respectively in $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$

$$\begin{aligned} \forall (x, \xi) \in \overline{\mathcal{O}}_1 \times \mathbb{R}^n; p_{\varphi_1}(x, \xi) = 0 &\implies \{\operatorname{Re}(p_{\varphi_1}), \operatorname{Im}(p_{\varphi_1})\}(x, \xi) > 0, \\ \forall (x, \xi) \in \overline{\mathcal{O}}_2 \times \mathbb{R}^n; p_{\varphi_2}(x, \xi) = 0 &\implies \{\operatorname{Re}(p_{\varphi_2}), \operatorname{Im}(p_{\varphi_2})\}(x, \xi) > 0. \end{aligned}$$

The Carleman estimate corresponding to the following transmission boundary value problem

$$(3.1) \quad \begin{cases} -\Delta w_1 - \frac{\alpha_1}{h} w_1 = f_1 & \text{in } \mathcal{O}_1 \\ -\Delta w_2 - \frac{\alpha_2}{h} w_2 = f_2 & \text{in } \mathcal{O}_2 \\ w_1 = w_2 + e_1 & \text{on } \gamma \\ \partial_\nu w_1 = \partial_\nu w_2 + e_2 & \text{on } \gamma \\ w_2 = 0 & \text{on } \gamma_2 \end{cases}$$

is given in the following

Theorem 3.1 [RR10, Theorem 2.1] *Under the above assumptions on the weight functions φ_1 and φ_2 , there exists $h_0 > 0$ and $C > 0$ such that*

$$(3.2) \quad \begin{aligned} & h \|e^{\varphi_1/h} w_1\|_{L^2(\mathcal{O}_1)}^2 + h^3 \|e^{\varphi_1/h} \nabla w_1\|_{L^2(\mathcal{O}_1)}^2 + h \|e^{\varphi_1/h} w_1\|_{L^2(\gamma)}^2 + h^3 \|e^{\varphi_1/h} \nabla w_1\|_{L^2(\gamma)}^2 + \\ & h^3 \|e^{\varphi_1/h} \partial_\nu w_1\|_{L^2(\gamma)}^2 + h \|e^{\varphi_2/h} w_2\|_{L^2(\mathcal{O}_2)}^2 + h^3 \|e^{\varphi_2/h} \nabla w_2\|_{L^2(\mathcal{O}_2)}^2 + h \|e^{\varphi_2/h} w_2\|_{L^2(\gamma)}^2 + \\ & h^3 \|e^{\varphi_2/h} \nabla w_2\|_{L^2(\gamma)}^2 + h^3 \|e^{\varphi_2/h} \partial_\nu w_2\|_{L^2(\gamma)}^2 \leq C(h^4 \|e^{\varphi_1/h} f_1\|_{L^2(\mathcal{O}_1)}^2 + h^4 \|e^{\varphi_2/h} f_2\|_{L^2(\mathcal{O}_2)}^2 + \\ & h \|e^{\varphi_1/h} w_1\|_{L^2(\gamma_1)}^2 + h^3 \|e^{\varphi_2/h} \partial_\nu w_1\|_{L^2(\gamma_1)}^2 + h \|e^{\varphi_1/h} e_1\|_{L^2(\gamma)}^2 + h^3 \|e^{\varphi_1/h} \nabla e_1\|_{L^2(\gamma)}^2 + h^3 \|e^{\varphi_1/h} e_2\|_{L^2(\gamma)}^2) \end{aligned}$$

for all $w_1 \in \mathcal{C}^\infty(\overline{\mathcal{O}_1})$ and $w_2 \in \mathcal{C}^\infty(\overline{\mathcal{O}_2})$ satisfying the system (3.1) and $h \in]0, h_0]$.

Remarks 3.1

- 1) If the function w_1 is supported away from γ_1 the estimate (3.2) is allows true even if we don't assume that $(\partial_\nu \varphi_1)|_{\gamma_1} \neq 0$, while the proof of Theorem 3.1 is local.
- 2) We can not assume that $(\partial_\nu \varphi_1)|_{\gamma_1} < 0$ (it means $\partial_\nu \varphi_1 < 0$ in whole $\partial \mathcal{O}_1$), otherwise the weight function attain his global maximum in \mathcal{O}_1 and thus our strategy of the construction of the phases is fails (see below).

3.2 Weight function's construction

In this section we will try to find two phases that satisfies the Hörmander's condition except in a finite number of ball where one of them do not satisfies this condition the second does and is strictly greater. Note that this result is similar to the Burq's one [Bur98, Proposition 3.2], but here we give a new proof due to F. Laudenbach. Then we will adapte this result to our case to constructe a suitable weight functions that will be needed in the following section. The main ingredient of this section is the following one.

Proposition 3.1 *Let \mathcal{O} be a bounded open subset with boundary $\gamma = \gamma_1 \cup \gamma_2$ where $\overline{\gamma_1} \cap \overline{\gamma_2} = \emptyset$, then there exists two real functions $\psi_1, \psi_2 \in \mathcal{C}^\infty(\mathcal{O})$ and continous on $\overline{\mathcal{O}}$ satisfying for $k = 1, 2$ that $(\partial_\nu \psi_k)|_{\gamma_1} < 0$ and $(\partial_\nu \psi_k)|_{\gamma_2} > 0$ having only degenerate critical points (of finite number) such that when $\nabla \psi_k = 0$ then $\nabla \psi_{\sigma(k)} \neq 0$ and $\psi_{\sigma(k)} > \psi_k$. Where σ is the permutation of the set $\{1, 2\}$ different from the identity.*

Remarks 3.2

- 1) One consequence of Proposition 3.1 is that for $k = 1, 2$ we can find a finite number of points x_{kj_k} and $j_k = 1, \dots, N_k$ and $\epsilon > 0$ such that $B(x_{kj_k}, 2\epsilon) \subset \overline{\mathcal{O}}$ and $B(x_{1j_1}, 2\epsilon) \cap B(x_{2j_2}, 2\epsilon) = \emptyset$, for all $k = 1, 2$ and $j_k = 1, \dots, N_k$ and in $B(x_{kj_k}, 2\epsilon)$ we have $\psi_{\sigma(k)} > \psi_k$ (See Figure 1).

- 2) For $\lambda > 0$ large enough the weight functions $\varphi_k = e^{\lambda \psi_k}$ satisfy the Hörmander's condition

in $U_k = \mathcal{O} \cap \left(\bigcup_{j_k=1}^{N_k} B(x_{kj_k}, \epsilon) \right)^c$. Indeed, we have only to prove that for an open bounded

subset $U \in \mathbb{R}^n$ and if $\psi \in \mathcal{C}^\infty(\overline{U})$ satisfying $|\nabla \psi| \geq C$ in \overline{U} and $\varphi = e^{\lambda \psi}$ we have $\{\text{Re}(p_\varphi), \text{Im}(p_\varphi)\}(x, \xi) \geq C'$ in $\overline{U} \times \mathbb{R}^n$ for $\lambda > 0$ large enough. We have

$$\begin{cases} \nabla \varphi = \lambda e^{\lambda \psi} \nabla \psi \text{ and } \varphi'' = e^{\lambda \psi} (\lambda \nabla \psi \cdot {}^t \nabla \psi + \lambda \psi'') \\ p_\varphi(x, \xi) = 0 \implies \langle \xi, \nabla \varphi \rangle = 0 \text{ and } |\xi|^2 = |\nabla \varphi|^2 \end{cases}$$

then we obtain

$$\begin{aligned} \{\operatorname{Re}(p_\varphi), \operatorname{Im}(p_\varphi)\}(x, \xi) &= 4\lambda e^{\lambda\psi} {}^t\xi \cdot \psi'' \cdot \xi + 4e^{3\lambda\psi} (\lambda^4 |\nabla\psi|^2 + \lambda^3 {}^t\nabla\psi \cdot \psi'' \cdot \nabla\psi) \\ &= 4e^{3\lambda\psi} (\lambda^4 |\nabla\psi|^2 + O(\lambda^3)). \end{aligned}$$

Which conclude the result.

- 3) In general, Proposition 3.1 is also true for any smooth manifold with boundary which the latter is the disjoint union of two open and closed submanifolds.

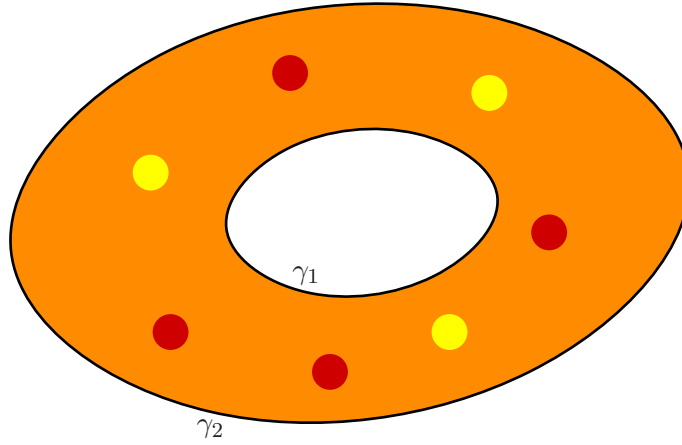


Figure 1: The domains of the weight functions φ_1 and ψ_1 (in yellow and orange), φ_2 and ψ_2 (in red and orange) where they have not critical points.

Proof :

While the Morse functions are dense (for the \mathcal{C}^∞ topology) in the set of \mathcal{C}^∞ functions then we can find ψ_1 a Morse function such that $(\partial_\nu \psi_1)|_{\gamma_1} < 0$ and $(\partial_\nu \psi_1)|_{\gamma_2} > 0$. We can suppose that ψ_1 have no local maximum in \mathcal{O} (The proceeding of the elimination of the maximum is described by Burq [Bur98, Appendix A], we can see also [Mil65, Theorem 8.1] and [Lau12, Lemma 2.6]).

Let c be a critical point of ψ_1 while its index is different from n then we can find a \mathcal{C}^∞ arc $\gamma_c : [-1, 1] \rightarrow \Omega$ such that $\gamma_c(0) = c$ and $\psi_1(\gamma_c(1)) = \psi_1(\gamma_c(-1)) > \psi_1(c)$. We do this construction for all the critical points of ψ_1 so that all the arcs are mutually disjoint. Hence, this allows us to find a vector field X in \mathcal{O} , vanishing near the boundary of \mathcal{O} such that for all critical points c of ψ_1 we have

$$X(\gamma_c(t)) = \dot{\gamma}_c(t),$$

where $\dot{\gamma}$ stand for the time derivative.

We denote ϕ_t its flow:

$$\dot{\phi}_t(x) = X(\phi_t(x)),$$

and we set $\psi_2 = \psi_1 \circ \phi_1$, thus ψ_1 and ψ_2 satisfy the required properties. Indeed, since $X \equiv 0$ near the boundary γ_1 and γ_2 which mean that $\phi_t(x) = x$ near γ_1 and γ_2 then $\partial_\nu \psi_1|_{\gamma_1} = \partial_\nu \psi_2|_{\gamma_1}$ and $\partial_\nu \psi_1|_{\gamma_2} = \partial_\nu \psi_2|_{\gamma_2}$. If c is a critical point of ψ_1 then we have $\psi_2(c) = \psi_1(\gamma_c(1)) > \psi_1(c)$, and if c' is a critical point of ψ_2 then $c' = \phi_{-1}(c)$ where c is a critical point of ψ_1 and we have $\psi_2(c') = \psi_1(\phi_1 \circ \phi_{-1}(c)) = \psi_1(c) < \psi_1(\phi_{-1}(c)) = \psi_1(c')$ by the construction of γ_c . ■

Now if we return to our geometric baseline as described in the introduction of this paper then according to Proposition 3.1 and Remark 3.2 and by noting $\tilde{\Omega}_1 = \Omega_1 \setminus \overline{B_r}$ where B_r is an open ball of Ω_1 with radius $r > 0$ such that $\overline{B_r} \subset \Omega_1$ we can find four phases $\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,1}$ and $\varphi_{2,2}$ verifying the Hörmander's condition respectively in $U_{1,1} = \tilde{\Omega}_1 \cap \left(\bigcup_{j=1}^{N_{11}} B(x_{11}^j, \epsilon) \right)^c$, $U_{1,2} = \tilde{\Omega}_1 \cap \left(\bigcup_{j_2=1}^{N_{12}} B(x_{12}^j, \epsilon) \right)^c$, $U_{2,1} = \Omega_2 \cap \left(\bigcup_{j_1=1}^{N_{21}} B(x_{21}^j, \epsilon) \right)^c$ and $U_{2,2} = \Omega_2 \cap \left(\bigcup_{j_2=1}^{N_{22}} B(x_{22}^j, \epsilon) \right)^c$ such that $|\nabla \varphi_{1,1}| > 0$ in $U_{1,1}$, $|\nabla \varphi_{1,2}| > 0$ in $U_{1,2}$, $|\nabla \varphi_{2,1}| > 0$ in $U_{2,1}$ and $|\nabla \varphi_{2,2}| > 0$ in $U_{2,2}$, moreover $\varphi_{1,k} < \varphi_{1,\sigma(k)}$ in $B(x_{1k}^j, 2\epsilon)$ for all $j = 1, \dots, N_{1,k}$ and $\varphi_{2,k} < \varphi_{2,\sigma(k)}$ in $B(x_{2k}^j, 2\epsilon)$ for all $j = 1, \dots, N_{2,k}$. Furthermore we have also for all $k = 1, 2$

$$(\partial_\nu \varphi_{1,k})|_S < 0, \quad (\partial_\nu \varphi_{2,k})|_S < 0 \text{ and } (\partial_\nu \varphi_{2,k})|_\Gamma < 0.$$

We can suppose also that $\varphi_{1,k}|_S = \varphi_{2,k}|_S$, and by argument of density we can suppose also that

$$(\partial_\nu \varphi_{1,k})|_S^2 - (\partial_\nu \varphi_{2,k})|_S^2 > 0.$$

And this concludes the construction of weight functions that will be used in next section.

4 Resolvent estimate

The purpose of this section is to find an estimate of the resolvent $(\mathcal{A} - i\mu \text{Id})^{-1}$ where μ is a real number such that $|\mu|$ is large enough. More precisely we prove that $\|(\mathcal{A} - i\mu \text{Id})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{C|\mu|}$ which imply the weak energy decay of the solution of the equation (1.1).

The main idea consiste to applying the Carleman estimates for a second order elliptic transmission system which is derived from the plate equation and this is what comes from the originality of our work, it means we prove the stability result for a system of fourth order by using an estimate of Carleman of second order only.

Let $(f_1, f_2, g_1, g_2) \in \mathcal{H}$ and $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ such that

$$(\mathcal{A} - i\mu \text{Id}) \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix},$$

then we get the following boundary value problem

$$(4.1) \quad \begin{cases} v_1 - i\mu u_1 = f_1 & \text{in } \Omega_1 \\ v_2 - i\mu u_2 = f_2 & \text{in } \Omega_2 \\ -\Delta(c_1^2 \Delta u_1 + a \Delta v_1) - i\mu v_1 = g_1 & \text{in } \Omega_1 \\ -c_2^2 \Delta^2 u_2 - i\mu v_2 = g_2 & \text{in } \Omega_2 \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \\ c_1 \Delta u_1 = c_2 \Delta u_2, \quad c_1 \partial_\nu \Delta u_1 = c_2 \partial_\nu \Delta u_2 & \text{on } S \\ u_2 = 0, \quad \Delta u_2 = 0 & \text{on } \Gamma. \end{cases}$$

Then the solution (u_1, u_2, v_1, v_2) of (4.1) satisfies

$$(4.2) \quad \begin{cases} v_1 = i\mu u_1 + f_1 & \text{in } \Omega_1 \\ v_2 = i\mu u_2 + f_2 & \text{in } \Omega_2 \\ \mu^2 u_1 - \Delta(c_1^2 \Delta u_1 + a \Delta v_1) = g_1 + i\mu f_1 & \text{in } \Omega_1 \\ \mu^2 u_2 - c_2^2 \Delta^2 u_2 = g_2 + i\mu f_2 & \text{in } \Omega_2 \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \\ c_1 \Delta u_1 = c_2 \Delta u_2, \quad c_1 \partial_\nu \Delta u_1 = c_2 \partial_\nu \Delta u_2 & \text{on } S \\ u_2 = 0, \quad \Delta u_2 = 0 & \text{on } \Gamma. \end{cases}$$

This can be rewritten as follows

$$(4.3) \quad \begin{cases} v_1 = i\mu u_1 + f_1 & \text{in } \Omega_1 \\ v_2 = i\mu u_2 + f_2 & \text{in } \Omega_2 \\ (-\Delta - \frac{|\mu|}{c_1})(c_1 \Delta u_1 + \frac{a}{c_1} \Delta v_1 - |\mu| u_1) = \Phi_1 = \frac{1}{c_1} g_1 + i \frac{\mu}{c_1} f_1 - a \frac{|\mu|}{c_1^2} \Delta v_1 & \text{in } \Omega_1 \\ (-\Delta - \frac{|\mu|}{c_2})(c_2 \Delta u_2 - |\mu| u_2) = \Phi_2 = \frac{1}{c_2} g_2 + i \frac{\mu}{c_2} f_2 & \text{in } \Omega_2 \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \\ c_1 \Delta u_1 = c_2 \Delta u_2, \quad c_1 \partial_\nu \Delta u_1 = c_2 \partial_\nu \Delta u_2 & \text{on } S \\ u_2 = 0, \quad \Delta u_2 = 0 & \text{on } \Gamma. \end{cases}$$

We set now

$$(4.4) \quad w_1 = c_1 \Delta u_1 - |\mu| u_1 + \frac{a}{c_1} \Delta v_1 \quad \text{and} \quad w_2 = c_2 \Delta u_2 - |\mu| u_2,$$

then it easy to show that w_1 and w_2 satisfy the following simple transmission problem

$$(4.5) \quad \begin{cases} -\Delta w_1 - \frac{|\mu|}{c_1} w_1 = \Phi_1 & \text{in } \Omega_1 \\ -\Delta w_2 - \frac{|\mu|}{c_2} w_2 = \Phi_2 & \text{in } \Omega_2 \\ w_1 = w_2, \quad \partial_\nu w_1 = \partial_\nu w_2 & \text{on } S \\ w_2 = 0 & \text{on } \Gamma. \end{cases}$$

We set also B_{4r} a ball of raduis $4r > 0$, such that $a(x) > 0$ in $B_{4r} \subset \omega$ and we recall the notation gived in the end of the previous section $\tilde{\Omega}_1 = \Omega_1 \setminus \overline{B_r}$. The most important ingredient of the proof of the resolvent estimate is the following lemma which is essentially a consequence of the Carleman estimate.

Lemma 4.1 *There exist a constant $C > 0$ such that for any $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ solution of (4.1) the following result holds*

$$(4.6) \quad \begin{aligned} & \|\Delta u_1\|_{L^2(\Omega_1)}^2 + \|\Delta u_2\|_{L^2(\Omega_2)}^2 + \|v_1\|_{L^2(\Omega_1)}^2 + \|v_2\|_{L^2(\Omega_2)}^2 \leq C e^{C|\mu|} \left(\|\Delta f_1\|_{L^2(\Omega_1)}^2 \right. \\ & \quad \left. + \|\Delta f_2\|_{L^2(\Omega_2)}^2 + \|g_1\|_{L^2(\Omega_1)}^2 + \|g_2\|_{L^2(\Omega_2)}^2 + \int_{\Omega_1} a |\Delta v_1|^2 dx + \int_{B_{4r}} |u_1|^2 dx \right), \end{aligned}$$

for all $\mu \in \mathbb{R}$ large enough.

Proof :

We introduce the cutt-off function $\chi \in \mathcal{C}^\infty(\Omega_1)$ by setting

$$\chi(x) = \begin{cases} 1 & \text{in } B_{3r}^c \\ 0 & \text{in } B_{2r} \end{cases}$$

Next, denote $\tilde{w}_1 = \chi w_1$. And by (4.5), one sees that

$$(4.7) \quad -\Delta \tilde{w}_1 - \frac{|\mu|}{c_1} \tilde{w}_1 = \tilde{\Phi}_1 = \chi \Phi_1 - [\Delta, \chi] w_1.$$

Now keeping the same notations as the previous section and let $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$ and $\varphi_{2,2}$ four weight functions that satisfies the conclusion of the section 3. Let $\chi_{1,1}$, $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{2,2}$ four cut-off functions equal to one respectively in $\left(\bigcup_{j=1}^{N_{11}} B(x_{1j}^1, 2\epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{12}} B(x_{1j}^2, 2\epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{21}} B(x_{2j}^1, 2\epsilon)\right)^c$ and $\left(\bigcup_{j=1}^{N_{22}} B(x_{2j}^2, 2\epsilon)\right)^c$ and supported respectively in $\left(\bigcup_{j=1}^{N_{11}} B(x_{1j}^1, \epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{12}} B(x_{1j}^2, \epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{21}} B(x_{2j}^1, \epsilon)\right)^c$ and $\left(\bigcup_{j=1}^{N_{22}} B(x_{2j}^2, \epsilon)\right)^c$ (in order to eliminate the critical points of the phases functions $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$ and $\varphi_{2,2}$ (See Figure 1)). We set now $w_{1,1} = \chi_{1,1} \tilde{w}_1$, $w_{1,2} = \chi_{1,2} \tilde{w}_1$, $w_{2,1} = \chi_{2,1} w_2$ and $w_{2,2} = \chi_{2,2} w_2$. Then from the system (4.5) for $k = 1, 2$ we obtain

$$(4.8) \quad \begin{cases} -\Delta w_{1,k} - \frac{|\mu|}{c_1} w_{1,k} = \Psi_{1,k} & \text{in } \Omega_1 \\ -\Delta w_{2,k} - \frac{|\mu|}{c_2} w_{2,k} = \Psi_{2,k} & \text{in } \Omega_2 \\ w_{1,k} = w_{2,k}, \quad \partial_\nu w_{1,k} = \partial_\nu w_{2,k} & \text{on } S \\ w_{2,k} = 0 & \text{on } \Gamma, \end{cases}$$

where

$$(4.9) \quad \begin{cases} \Psi_{1,k} = \chi_{1,k} \tilde{\Phi}_1 - [\Delta, \chi_{1,k}] \tilde{w}_1 \\ \Psi_{2,k} = \chi_{2,k} \Phi_2 - [\Delta, \chi_{2,k}] w_2. \end{cases}$$

Applying now the Carleman estimate given in the previous section (Theorem 3.1) to the system (4.8) for $h = \frac{1}{|\mu|}$ then for $k = 1, 2$ we obtain

$$h \|e^{\varphi_{1,k}/h} w_{1,k}\|_{L^2(U_{1,k})}^2 + h^3 \|e^{\varphi_{1,k}/h} \nabla w_{1,k}\|_{L^2(U_{1,k})}^2 + h \|e^{\varphi_{2,k}/h} w_{2,k}\|_{L^2(U_{2,k})}^2 + h^3 \|e^{\varphi_{2,k}/h} \nabla w_{2,k}\|_{L^2(U_{2,k})}^2 \leq Ch^4 (\|e^{\varphi_{1,k}/h} \Psi_{1,k}\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{2,k}/h} \Psi_{2,k}\|_{L^2(U_{2,k})}^2).$$

Relations (4.7) and (4.9) yields

$$(4.10) \quad \begin{aligned} & h \|e^{\varphi_{1,k}/h} w_{1,k}\|_{L^2(U_{1,k})}^2 + h^3 \|e^{\varphi_{1,k}/h} \nabla w_{1,k}\|_{L^2(U_{1,k})}^2 + h \|e^{\varphi_{2,k}/h} w_{2,k}\|_{L^2(U_{2,k})}^2 + \\ & h^3 \|e^{\varphi_{2,k}/h} \nabla w_{2,k}\|_{L^2(U_{2,k})}^2 \leq Ch^4 (\|e^{\varphi_{1,k}/h} \Phi_1\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{2,k}/h} \Phi_2\|_{L^2(U_{2,k})}^2 + \\ & \|e^{\varphi_{1,k}/h} [\Delta, \chi] w_1\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{1,k}/h} [\Delta, \chi_{1,k}] \tilde{w}_1\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{2,k}/h} [\Delta, \chi_{2,k}] w_2\|_{L^2(U_{2,k})}^2). \end{aligned}$$

We addition the two last estimates for $k = 1, 2$ and using the properties of phases $\varphi_{1,k} < \varphi_{1,\sigma(k)}$ in $\left(\bigcup_{j=1}^{N_{1k}} B(x_{1k}^j, 2\epsilon)\right)$ and $\varphi_{2,k} < \varphi_{2,\sigma(k)}$ in $\left(\bigcup_{j=1}^{N_{2k}} B(x_{2k}^j, 2\epsilon)\right)$ then we can absorb the terms $[\Delta, \chi_{1,k}]\tilde{w}_1$ and $[\Delta, \chi_{2,k}]w_2$ at the right hand side of (4.10) into the left hand side for $h > 0$ small. More precisely we obtain

$$\begin{aligned} & h \int_{\tilde{\Omega}_1} (e^{2\varphi_{1,1}/h} + e^{2\varphi_{1,2}/h}) |\tilde{w}_1|^2 dx + h \int_{\Omega_2} (e^{2\varphi_{2,1}/h} + e^{2\varphi_{2,2}/h}) |w_2|^2 dx \leq \\ & Ch^4 \left(\int_{\Omega_1} (e^{2\varphi_{1,1}/h} + e^{2\varphi_{1,2}/h}) |\Phi_1|^2 dx + \int_{\Omega_2} (e^{2\varphi_{2,1}/h} + e^{2\varphi_{2,2}/h}) |\Phi_2|^2 dx \right. \\ & \quad \left. + \int_{\tilde{\Omega}_1} (e^{2\varphi_{1,1}/h} + e^{2\varphi_{1,2}/h}) |[\Delta, \chi]w_1|^2 dx \right). \end{aligned}$$

Consequently, by using that $\Omega_1 = \tilde{\Omega}_1 \cup B_{2r}$ and the expressions of Φ_1 and Φ_2 in (4.3) we see that

$$\begin{aligned} (4.11) \quad & \int_{\Omega_1} |w_1|^2 dx + \int_{\Omega_2} |w_2|^2 dx \leq Ce^{C/h} \left(\int_{\Omega_1} |f_1|^2 dx + \int_{\Omega_1} |g_1|^2 dx + \int_{\Omega_2} |f_2|^2 dx \right. \\ & \quad \left. + \int_{\Omega_2} |g_2|^2 dx + \int_{\Omega_1} a |\Delta v_1|^2 dx + \int_{B_{2r}} |w_1|^2 dx + \int_{\tilde{\Omega}_1} |[\Delta, \chi]w_1|^2 dx \right). \end{aligned}$$

To accomplish the proof of the lemma we estimate the two last terms in the right hand side of (4.11). We set $\tilde{\chi}$ a cutt-off function equal to 1 in a neighborhood of B_{3r} and supported in B_{4r} then we have

$$(-1 + \Delta)(\tilde{\chi}w_1) = [\Delta, \tilde{\chi}]w_1 - \tilde{\chi}w_1 - \frac{|\mu|}{c_1} \tilde{\chi}w_1 - \tilde{\chi}\Phi_1,$$

and hence by elliptic estimates (see [WRL95]) we get

$$\begin{aligned} (4.12) \quad & \|w_1\|_{H^1(B_{3r})}^2 \leq C(\|(-1 + \Delta)(\tilde{\chi}w_1)\|_{H^{-1}(B_{4r})}^2 + \|w_1\|_{L^2(B_{4r})}^2) \\ & \leq C(\|\Phi_1\|_{L^2(\Omega_1)}^2 + (1 + |\mu|^2)\|w_1\|_{L^2(B_{4r})}^2) \\ & \leq C \left(|\mu|^2 \|f_1\|_{L^2(\Omega_1)}^2 + \|g_1\|_{L^2(\Omega_1)}^2 + (1 + |\mu|^2)\|w_1\|_{L^2(B_{4r})}^2 + |\mu|^2 \int_{\Omega_1} a |\Delta v_1|^2 dx \right). \end{aligned}$$

Since $\text{supp}([\Delta, \chi]) \subset B_{3r}$ we deduce from (4.4) and (4.12) that

$$\begin{aligned} (4.13) \quad & \int_{B_{2r}} |w_1|^2 dx + \int_{\tilde{\Omega}_1} |[\Delta, \chi]w_1|^2 dx \leq C\|w_1\|_{H^1(B_{3r})}^2 \\ & \leq C \left(|\mu|^2 \|f_1\|_{L^2(\Omega_1)}^2 + \|g_1\|_{L^2(\Omega_1)}^2 + (1 + |\mu|^2)^2 \|u_1\|_{L^2(B_{4r})}^2 + |\mu|^2 \int_{\Omega_1} a |\Delta v_1|^2 dx \right). \end{aligned}$$

On other hand from (4.4) and the transmission conditions we see that

$$\begin{aligned} (4.14) \quad & \|w_1\|_{L^2(\Omega_1)}^2 + \|w_2\|_{L^2(\Omega_2)}^2 \geq \|c_1 \Delta u_1 - |\mu|u_1\|_{L^2(\Omega_1)}^2 + \|c_2 \Delta u_2 - |\mu|u_2\|_{L^2(\Omega_2)}^2 - C \int_{\Omega_1} a |\Delta v_1|^2 dx \\ & \geq -C \int_{\Omega_1} a |\Delta v_1|^2 dx + c_1^2 \|\Delta u_1\|_{L^2(\Omega_1)}^2 + c_2^2 \|\Delta u_2\|_{L^2(\Omega_2)}^2 + |\mu|^2 (\|u_1\|_{L^2(\Omega_1)}^2 + \|u_2\|_{L^2(\Omega_2)}^2) \\ & \quad + |\mu| (\|\nabla u_1\|_{L^2(\Omega_1)}^2 + \|\nabla u_2\|_{L^2(\Omega_2)}^2) \geq \|\Delta u_1\|_{L^2(\Omega_1)}^2 + \|\Delta u_2\|_{L^2(\Omega_2)}^2 - C \int_{\Omega_1} a |\Delta v_1|^2 dx, \end{aligned}$$

and by the expression of v_1 and v_2 in (4.2) we obtain

$$(4.15) \quad \begin{aligned} \|v_1\|_{L^2(\Omega_1)}^2 &\leq \|f_1\|_{L^2(\Omega_1)}^2 + |\mu|^2 \|u_1\|_{L^2(\Omega_1)}^2 \\ \|v_2\|_{L^2(\Omega_2)}^2 &\leq \|f_2\|_{L^2(\Omega_2)}^2 + |\mu|^2 \|u_2\|_{L^2(\Omega_2)}^2. \end{aligned}$$

Then by combining Proposition 2.2, and estimates (4.11), (4.13), (4.14) and (4.15) we obtain the results. \blacksquare

At this step we suppose now that the resolvent estimate (1.3) is not true. Then there exist $K_m > 0$, $\mu_m \in \mathbb{R}$ and a two families $(u_{1,m}, u_{2,m}, v_{1,m}, v_{2,m}) \in \mathcal{D}(\mathcal{A})$ and $(f_{1,m}, f_{2,m}, g_{1,m}, g_{2,m}) \in \mathcal{H}$, $m = 1, 2, \dots$ such that

$$(4.16) \quad |\mu_m| \longrightarrow +\infty, \quad K_m \longrightarrow +\infty, \quad \|(u_{1,m}, u_{2,m}, v_{1,m}, v_{2,m})\|_{\mathcal{H}} = 1,$$

and

$$(4.17) \quad e^{K_m |\mu_m|} (\mathcal{A} - i\mu_m) \begin{pmatrix} u_{1,m} \\ u_{2,m} \\ v_{1,m} \\ v_{2,m} \end{pmatrix} = \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ g_{1,m} \\ g_{2,m} \end{pmatrix} \longrightarrow 0 \text{ in } \mathcal{H}.$$

This imply that

$$(4.18) \quad e^{K_m |\mu_m|} (v_{1,m} - i\mu_m u_{1,m}) = f_{1,m} \longrightarrow 0 \text{ in } H^2(\Omega_1),$$

$$(4.19) \quad e^{K_m |\mu_m|} (v_{2,m} - i\mu_m u_{2,m}) = f_{2,m} \longrightarrow 0 \text{ in } H^2(\Omega_2),$$

$$(4.20) \quad e^{K_m |\mu_m|} (-\Delta(c_1^2 \Delta u_{1,m} + a \Delta v_{1,m}) - i\mu_m v_{1,m}) = g_{1,m} \longrightarrow 0 \text{ in } L^2(\Omega_1),$$

$$(4.21) \quad e^{K_m |\mu_m|} (-c_2^2 \Delta^2 u_{2,m} - i\mu_m v_{2,m}) = g_{2,m} \longrightarrow 0 \text{ in } L^2(\Omega_2).$$

From (4.16) and (4.17), we get

$$(4.22) \quad \operatorname{Re} \left\langle \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ g_{1,m} \\ g_{2,m} \end{pmatrix}, \begin{pmatrix} u_{1,m} \\ u_{2,m} \\ v_{1,m} \\ v_{2,m} \end{pmatrix} \right\rangle_{\mathcal{H}} = -e^{K_m |\mu_m|} \int_{\Omega_1} a |\Delta v_{1,m}|^2 dx \longrightarrow 0.$$

Then by (4.18) and (4.22), we obtain

$$(4.23) \quad |\mu_m|^2 e^{\frac{K_m}{2} |\mu_m|} \int_{\omega} |\Delta u_{1,m}|^2 dx \longrightarrow 0.$$

Hence from (4.22) and (4.23) we obtain

$$(4.24) \quad e^{\frac{K_m}{2} |\mu_m|} \left(\int_{\omega} |\Delta u_{1,m}|^2 dx + \int_{\omega} |\Delta v_{1,m}|^2 dx \right) \longrightarrow 0.$$

And by (4.18) we have

$$(4.25) \quad \frac{1}{|\mu_m|^2} \|\Delta(\psi \cdot v_{1,m})\|_{L^2(\Omega_1)}^2 = O(1), \quad \forall \psi \in \mathcal{C}^\infty(\Omega_1).$$

Then by multiplying (4.20) by $\mu_m^{-1}\psi \cdot \bar{v}_{1,m}$ where $\psi \in \mathcal{C}^\infty(\Omega_1)$ and $\text{supp}(\psi) \subset \omega$ we obtain by (4.24) and (4.25) that

$$e^{\frac{Km}{4}|\mu_m|} \int_{\omega} |v_{1,m}|^2 \psi \, dx \longrightarrow 0.$$

In particular we obtain that

$$e^{\frac{Km}{4}|\mu_m|} \int_{B_{4r}} |v_{1,m}|^2 \, dx \longrightarrow 0.$$

Then also we get by (4.18) that

$$(4.26) \quad e^{\frac{Km}{4}|\mu_m|} \int_{B_{4r}} |u_{1,m}|^2 \, dx \longrightarrow 0.$$

Now by applying inequality (4.6) to the system (4.18)-(4.21) it follows that

$$(4.27) \quad \begin{aligned} & \|\Delta u_{1,m}\|_{L^2(\Omega_1)}^2 + \|\Delta u_{2,m}\|_{L^2(\Omega_2)}^2 + \|v_{1,m}\|_{L^2(\Omega_1)}^2 + \|v_{2,m}\|_{L^2(\Omega_2)}^2 \leq \\ & C e^{C|\mu_m|} \left(e^{-2Km|\mu_m|} \left(\|\Delta f_{1,m}\|_{L^2(\Omega_1)}^2 + \|\Delta f_{2,m}\|_{L^2(\Omega_2)}^2 + \|g_{1,m}\|_{L^2(\Omega_1)}^2 + \|g_{2,m}\|_{L^2(\Omega_2)}^2 \right) \right. \\ & \quad \left. + e^{-\frac{Km}{4}|\mu_m|} \left(\int_{\Omega_1} a |\Delta v_{1,m}|^2 \, dx + \int_{B_{4r}} |u_{1,m}|^2 \, dx \right) e^{\frac{Km}{4}|\mu_m|} \right). \end{aligned}$$

While the right hand side of (4.27) go to zero as $m \longrightarrow +\infty$ by (4.16)-(4.17) and estimates (4.22) and (4.26), then we obtain a contradiction with (4.16). And this conclude the proof of the resolvent estimate.

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